

# Magnetic field decay with Hall drift in neutron star crusts

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## ABSTRACT

The dynamics of magnetic field decay with Hall drift is investigated. Assuming that axisymmetric magnetic fields are located in a spherical crust with uniform conductivity and electron number density, long-term evolution is calculated up to Ohmic dissipation. The nonlinear coupling between poloidal and toroidal components is explored in terms of their energies and helicity. Nonlinear oscillation by the drift in strongly magnetized regimes is clear only around the equipartition between two components. Significant energy is transferred to the poloidal component when the toroidal component initially dominates. However, the reverse is not true. Once the toroidal field is less dominant, it quickly decouples due to a larger damping rate. The polar field at the surface is highly distorted from the initial dipole during the Hall drift timescale, but returns to the initial dipole in a longer dissipation timescale, since it is the least damped one.

**Key words:** stars:neutron—stars:magnetars—magnetic fields

## 1 INTRODUCTION

The recent discovery of soft gamma repeaters (SGRs) with weak fields (Rea et al. 2010) has again raised problems related to magnetic field evolution in isolated neutron stars. The activity of magnetars with SGRs and anomalous X-ray pulsars (AXPs) has been believed to be powered by the decay of ultrastrong magnetic fields (Thompson & Duncan 1995, 1996). Their surface dipole field, which is observationally inferred from the spin period and its time derivative, typically exceeds electron quantum magnetic field  $B_Q = 4.4 \times 10^{13}$  G. The field strength of SGR0418+5729 is, however, relatively weak at  $< 7.5 \times 10^{12}$  G (Rea et al. 2010). The critical boundary between magnetars and radio pulsars has thus become less clear, and magnetars are not sufficiently characterized by their dipole field strength alone. Their activity may be explained by hidden magnetic fields such as poloidal components with higher-order multipoles or internal toroidal components. In either case, the field strength should be greater than  $B = 10^{14}$ – $10^{15}$  G, since other energy sources are insufficient (see, for example, the review by Mereghetti (2008)).

The importance of Hall drift on the magnetic evolution of neutron stars was pointed out before evidence of the existence of magnetars was available (Jones 1988; Goldreich & Reisenegger 1992). The effect, which depends on the field strength, becomes more important in strong regimes where  $B > 10^{14}$  G. Furthermore, Hall drift may induce instability under certain conditions (Rheinhardt & Geppert 2002). The effect has been considered in analytic treatments (Cumming et al. 2004; Reisenegger et al. 2007), and as plane-parallel slab geometry (Vainshtein et al. 2000; Geppert et al. 2003; Rheinhardt et al. 2004). These studies are useful to understand some aspects of the mechanism, but nonlinear numerical analysis is also required to examine the behavior in more realistic stars. Several simulations have also been performed, assuming that the fields are located in the spherical crust region of a neutron star (Naito & Kojima 1994; Shalybkov & Urpin 1997; Hollerbach & Rüdiger 2002, 2004; Pons & Geppert 2007). In particular, Hollerbach & Rüdiger (2002, 2004) extensively studied the effect in a crust with uniform conductivity and density, and subsequently extended the discussion to stratified stellar models. Pons & Geppert (2007) also calculated the evolution in a realistic stratified model with a thermal history. These numerical calculations are based on spectral or quasi-spectral methods, for example expanding angular functions using spherical harmonics. Limitations of such an approach are discussed therein. For example, evolution of a purely toroidal field forms a steep gradient that cannot be

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calculated (Pons & Geppert 2007). A recent promising approach is using a finite difference scheme to examine the nonlinear evolution of the Hall instability in a 2D slab (Pons & Geppert 2010). Each numerical scheme has advantages and disadvantages, so multiple complementary approaches are needed.

In this paper, we calculate magnetic field evolution using a finite difference scheme to understand nonlinear Hall drift dynamics. The model is simplified by assuming an axisymmetric magnetic field located in a crust with uniform conductivity and electron number density. This paper is organized as follows. In section 2, the model and its assumptions are described. Magnetic field evolution is governed by an induction equation with nonlinear Hall drift term. The relevant boundary conditions and initial configurations are also discussed. Sections 3 and 4 give numerical results for two distinct initial configurations. One condition is a purely toroidal field in which the poloidal part is always zero if it is exactly zero at the initial state. This is in contrast to a purely poloidal initial case, for which the toroidal part is inevitably induced. The evolution of a purely toroidal field is furthermore of interest, due to the similarity to Burgers' equation (e.g., Whitham (1974)). Vainshtein et al. (2000) study the problem in a stratified plane-parallel slab, which corresponds to small-scale dynamics much smaller than the stellar radius. The results clarify the local mechanism, but our concern is global aspects. How do the results change in a spherical shell like the crust? This problem is numerically studied in section 3. Section 4 describes a second study related to the evolution of mixed fields with poloidal and toroidal components. Magnetohydrodynamics simulation (Braithwaite & Nordlund 2006; Braithwaite & Spruit 2006; Braithwaite 2009) shows that dynamically stable configurations are such mixed ones, in which poloidal and toroidal field strengths are of the same order. There is little known about the initial configuration of neutron stars in particular, the location, topology, and field strength. If both components coexist, their strengths are likely to be similar. It is therefore important to explore energy transfer by Hall drift between the components over long-term evolution through Ohmic dissipation. Section 5 presents our conclusions.

## 2 MODEL AND FORMULATION

### 2.1 Equations

The magnetic field evolution is governed by the induction equation with the Hall term

$$\frac{\partial}{\partial t} \vec{B} = -\vec{\nabla} \times \left( \frac{c^2}{4\pi\sigma} \vec{\nabla} \times \vec{B} \right) + \vec{\nabla} \times \left[ \frac{c}{4\pi en_e} \vec{B} \times (\vec{\nabla} \times \vec{B}) \right], \quad (1)$$

where  $\sigma$  is the electric conductivity,  $n_e$  the electron number density,  $e$  the charge density, and  $c$  the speed of light (Goldreich & Reisenegger 1992; Naito & Kojima 1994). There are two typical timescales associated with the first and second terms in eq. (1), the Ohmic decay timescale  $4\pi\sigma L^2/c^2$  and the Hall drift timescale  $4\pi en_e L^2/(cB_0)$ . Here,  $L$  and  $B_0$  are typical values for the spatial length and magnetic field strength. In general,  $\sigma$  and  $n_e$  depend on the spatial position and the time through the thermal history. In this paper, however, for simplicity they are assumed to be constant to explore the dynamics. Thus, the behavior of this system is specified by the ratio of two timescales,  $\mathcal{R}_m = \sigma B_0/(en_e c)$ , which is called the magnetized parameter by Pons & Geppert (2007). The parameter  $\mathcal{R}_m$  is given by the initial maximum value of the magnetic field. The typical value for a neutron star is  $\mathcal{R}_m = 1\text{--}10 \times (B_0/10^{13}\text{G})$  (Pons & Geppert 2007), but this significantly depends on spatial position and temperature (Cumming et al. 2004). The overall physical timescale is scaled by  $\tau_d = 4\pi\sigma r_s^2/c^2$ , where the stellar radius  $r_s$  is used for normalization. Note that the characteristic decay timescale is  $\tau_d$  for node-less field filled in a sphere, but it becomes smaller for the field localized in a crust. A magnetic field with  $\mathcal{R}_m(>1)$  is initially given, and the evolution is numerically followed until decay at  $\sim \tau_d$ . In the case where  $\mathcal{R}_m \gg 1$ , which is relevant to strong fields like those of magnetars, the second term on the right-hand side of eq. (1) dominates. The advection term is nonlinear, and treating it becomes complicated.

Magnetic fields with axial symmetry ( $\partial/\partial\phi = 0$ ) are described by two functions, a flux function  $G$  describing the poloidal magnetic field and a stream function  $S$  describing poloidal current flow:

$$\vec{B} = \frac{1}{R} (\vec{\nabla} G \times \vec{e}_\phi) + \frac{S}{R} \vec{e}_\phi, \quad (2)$$

where  $R = r \sin \theta$  is the cylindrical radius in spherical coordinates  $(r, \theta, \phi)$ . Ampere's equation gives the current density as

$$\frac{4\pi}{c} \vec{j} = \vec{\nabla} \times \vec{B} = \frac{1}{R} (\vec{\nabla} S \times \vec{e}_\phi) - \frac{1}{R} \mathcal{D}(G) \vec{e}_\phi, \quad (3)$$

where  $\mathcal{D}(G)$  is given by

$$\mathcal{D}(G) = \left( \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) G, \quad (4)$$

or in cylindrical coordinates  $(R, Z, \phi)$  by

$$\mathcal{D}(G) = \left( R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2} \right) G. \quad (5)$$

The magnetic field evolution (1) is written in terms of  $G$  and  $S$  (e.g., Reisenegger et al. (2007)) as

$$\frac{\partial G}{\partial t} = \frac{1}{\tau_d} \mathcal{D}(G) + \frac{\mathcal{R}_m}{\tau_d R} (\vec{\nabla} G \times \vec{\nabla} S) \cdot \vec{e}_\phi, \quad (6)$$

$$\frac{\partial S}{\partial t} = \frac{1}{\tau_d} \mathcal{D}(S) + \frac{\mathcal{R}_m R}{\tau_d} \left[ \vec{\nabla} \times \left\{ \frac{1}{R^2} (\mathcal{D}(G) \vec{\nabla} G + S \vec{\nabla} S) \right\} \right] \cdot \vec{e}_\phi, \quad (7)$$

where  $G$ ,  $S$ , and the spatial length are appropriately normalized. As has been pointed out (e.g., Vainshtein et al. (2000); Pons & Geppert (2007)), the evolutionary equation of a purely toroidal magnetic field is very similar to Burgers' equation, which is a simple example of nonlinear propagation with diffusion in one spatial dimension (Whitham 1974). Equation (7) for  $G = 0$  is thus reduced to

$$\frac{\partial S}{\partial t} = \frac{1}{\tau_d} \left( R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2} \right) S + \frac{2\mathcal{R}_m S}{\tau_d R^2} \frac{\partial S}{\partial Z}. \quad (8)$$

If the function  $S$  depends only on  $Z$ , eq. (8) is exactly Burgers' equation.

The functions  $G$  and  $S$  in eq. (2) can be expressed by a sum of Legendre polynomials  $P_l(\theta)$ :

$$G = - \sum_{l=1} g_l(r, t) \sin \theta \frac{\partial P_l(\theta)}{\partial \theta}, \quad (9)$$

$$S = - \sum_{l=1} s_l(r, t) \sin \theta \frac{\partial P_l(\theta)}{\partial \theta}. \quad (10)$$

The functions  $g_l$  and  $s_l$  independently evolve in the absence of the Hall term ( $\mathcal{R}_m = 0$ ). However, nonlinear coupling among  $g_l$  and  $s_l$  with different indices  $l$  becomes important with the increase of  $\mathcal{R}_m$ . Most numerical calculations of magnetic decay with the Hall effect have been performed by such an expansion (Naito & Kojima 1994; Hollerbach & Rüdiger 2002, 2004; Pons & Geppert 2007), with an exception (Shalybkov & Urpin 1997). Angular part of the field is expanded by spherical harmonics, but finite difference is used for radial direction in quasi-spectral method (Naito & Kojima 1994; Pons & Geppert 2007). Tchebycheff polynomials are used for radial direction in spectral method (Hollerbach & Rüdiger 2002, 2004). The limitation of spectral or quasi-spectral methods is also discussed, for example by Pons & Geppert (2007). A numerical Gibbs oscillation appears when the function evolves to form a steep gradient caused by advection. This is likely to occur in the large magnetized parameter  $\mathcal{R}_m$ . The finite difference method is used in this paper as an alternative approach. The numerical scheme used is a simple stable one, first-order forward time differencing and second-order centered space (FTCS) differencing (see, for example, Press et al. (1992)). There are more sophisticated schemes, but it is not easy to apply them to the nonlinear Hall drift term, which is most important here. The grid is staggered one with equal spacing. Typical number of grid points is  $100 \times 120$ . Our numerical results are verified using previous results based on the spectral method in Hollerbach & Rüdiger (2002), and the method works well for most parameters.

## 2.2 Initial configuration and boundary conditions

The magnetic field is assumed to be located outside the superconducting core. The crust ranges from  $r_1$  to the surface  $r_s$ . A typical size in neutron star models is  $(r_s - r_1)/r_s \sim 0.1$ , and depends on the equation of state and the stellar mass. A slightly thicker crust model is chosen here,  $r_1/r_s = 0.75$ , because this allows an easier demonstration of the numerical results. In actual implementation it is necessary to use more realistic models that include stratified conductivity, number density, and so on, but this simple model is useful for understanding the fundamental dynamics.

The magnetic field cannot penetrate into the core ( $r < r_1$ ). The condition for the toroidal field is simply  $S = 0$  at  $r_1$ . The condition for the poloidal field means that  $G$  is a constant at  $r_1$ , which is chosen as  $G = 0$ . As discussed in Hollerbach & Rüdiger (2002); Pons & Geppert (2007), the tangential components of the electric field,  $E_\theta$  and  $E_\phi$ , should vanish. They are explicitly given by

$$E_{(\theta, \phi)} = \frac{1}{\sigma} j_{(\theta, \phi)} + \frac{1}{en_e} (\vec{j} \times \vec{B})_{(\theta, \phi)}. \quad (11)$$

The second term on the right-hand side vanishes because  $\vec{B} = 0$  at  $r_1$ . The first term may be negligible for a large conductivity  $\sigma$ . Otherwise, some conditions should be imposed to cause the tangential current to vanish. The condition  $j_\theta = 0$  is satisfied if the function  $S$  rapidly approaches 0 as  $r$  goes to  $r_1$ . The condition  $j_\phi = 0$  should be imposed on the function  $G$ , more precisely on  $\mathcal{D}(G)$ , which is not easily treated. In the numerical simulation, the conditions  $G = S = 0$  are used at  $r_1$ , assuming a large  $\sigma$ . The current distributions should be set up at the initial time for the conditions to be satisfied.

The exterior of the star is assumed to be vacuum. The toroidal field should vanish, so that the boundary condition is  $S = 0$  at  $r_s$ . The vacuum solution of the poloidal field is expressed by a sum of multipole fields,

$$G = - \sum_{l=1} \frac{a_l}{r^l} \sin \theta \frac{\partial P_l(\theta)}{\partial \theta}, \quad (12)$$

where the coefficients  $a_l$  represent the multipole moments. For example, the dipole moment is  $a_1 = \mu$ . The radial component  $B_r$  should be continuous to the exterior vacuum solution across the surface by the condition  $\vec{\nabla} \cdot \vec{B} = 0$ , while  $B_\theta$  may be discontinuous if a surface current is allowed. In this paper, neglecting the surface currents, both components are assumed to be continuous, and the coefficients  $a_l$  are calculated by the interior numerical function  $G$  at the surface  $r = r_s$ . The surface boundary condition can be expressed by a sum of  $a_l$  up to  $l_{max} = 20-30$ . The results slightly change if the truncation is  $l_{max} < 10$ , but do not change even if  $l_{max}$  is further increased. Compared with spectral or quasi-spectral methods, small scale structure is not evident in our finite difference method. Initial configuration contains  $l=1$  or 2 component, so that the truncation is justified. Another approach without the multipole expansion is proposed as non-local boundary condition by Green's formula (Pons & Geppert 2010), but is not used here. Finally, for the boundary condition on the symmetric axis,  $\theta = 0$  and  $\pi$  are the regularity for the functions  $G$  and  $S$ , respectively. In other words,  $G = 0$  and  $S = 0$  there.

The functions  $G$  and  $S$  at the initial time of the numerical simulation should also be subject to the boundary conditions. The configuration is easily specified by the functions  $g_l$  and  $s_l$  in eqs. (9) and (10). Initial current distribution is chosen as  $\vec{j} = 0$  at both inner and outer boundaries. One simple solution satisfying this is given by

$$s_l(r, 0) = b_l \sin^2 \left( \frac{\pi(r - r_1)}{r_s - r_1} \right), \quad (13)$$

where  $b_l$  is a constant. Both the function  $s_l$  and its derivative  $ds_l/dr$  approach 0 as  $r \rightarrow r_1$  and  $r \rightarrow r_s$ . For the poloidal field, the function  $g_l(r, 0)$  is numerically given by solving

$$\left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) g_l = - \sin \left( \frac{\pi(r - r_1)}{r_s - r_1} \right), \quad (14)$$

with boundary conditions  $g_l(r_1, 0) = 0$  and  $g_l(r_s, 0) = a_l$ . The source term in eq. (14) comes from  $j_\phi$ , which is chosen to be localized near the geometrical center  $r \approx (r_1 + r_s)/2$ . The function  $g_l(r, 0)$  is smoothly connected with the multipole solution  $a_l/r^l$  at the surface  $r_s$ .

### 2.3 Energy and helicity

In previous works (e.g., Hollerbach & Rüdiger (2002, 2004)), coefficients of Legendre polynomials were utilized to study the system dynamics, and are useful to understand, for example, energy transfer among different wavelengths at a given time. In order to examine the whole dynamics, it is necessary to calculate the coefficients for many times, because they are time-dependent. Our concern in this paper, however, is global aspects, so that energy  $E(t)$  and helicity  $H(t)$  are used to represent the magnetic fields. These are indicators of the field strength and twisted structure. Integrating over the entire space, they are given by

$$E = \frac{1}{8\pi} \int (\vec{B} \cdot \vec{B}) dV, \quad (15)$$

$$H = \int (\vec{A} \cdot \vec{B}) dV, \quad (16)$$

where  $dV$  is the three-dimensional volume element, and  $\vec{A}$  is a vector potential for  $\vec{B}$ . The energy is divided to poloidal  $E_P$  and toroidal  $E_T$  parts,  $E = E_P + E_T$ , by which the energy transfer between them is examined. From eq. (2), the explicit forms are given by functions  $G$  and  $S$ :

$$E_P = \frac{1}{8\pi} \int \left( \frac{\vec{\nabla} G}{R} \right)^2 dV, \quad E_T = \frac{1}{8\pi} \int \left( \frac{S}{R} \right)^2 dV, \quad H = 2 \int \frac{GS}{R^2} dV. \quad (17)$$

From eqs. (1) and (3), the evolution of magnetic energy is given by

$$\frac{d}{dt} E = - \int \frac{1}{\sigma} (\vec{j} \cdot \vec{j}) dV. \quad (18)$$

This is nothing but Ohmic dissipation. The Hall drift does not concern the energy dissipation (Goldreich & Reisenegger 1992; Naito & Kojima 1994), but may affect the dissipation rate by modifying the current distribution. Similarly, the evolution of magnetic helicity  $H$  is

$$\frac{d}{dt} H = - \int 2c(\vec{E} \cdot \vec{B}) dV = - \int \frac{2c}{\sigma} (\vec{j} \cdot \vec{B}) dV. \quad (19)$$

In this way, both energy and helicity decay through Ohmic dissipation. Note that  $E$  and its dissipation term, the right-hand side of eq. (18), have a definite sign, but  $H$  and its dissipation term, the right-hand side of eq. (19) do not. The energy should monotonically decrease, but evolution of the helicity is unknown without numerical integration. In the numerical calculations, the energy balance equation (18) is used to check the accuracy. The relative errors to the initial magnetic energy are less than  $10^{-2}$ . They decrease with increase of the grid number. Relatively large numerical error is produced in the simulation with large magnetized parameter  $\mathcal{R}_m$ . It is also prominent in the initial evolution, in which steep structure is formed. Functions in the late phase are rather smooth, so that the errors do not accumulate so much, even if numerical integration continues to a longer time.

### 3 EVOLUTION OF PURELY TOROIDAL FIELDS

This section discusses the evolution of purely toroidal magnetic fields. The system is similar to that of Burgers' equation, as discussed in Section 2. The initial configuration of  $S$  is specified by a single component  $s_l(r, 0)$  in eq. (13). The three models considered here are characterized by  $l = 1$  with  $b_1 > 0$  in Model A,  $l = 2$  with  $b_2 > 0$  in Model B, and  $l = 2$  with  $b_2 < 0$  in Model C. The toroidal magnetic field  $B_\phi$  in Model C has the same amplitude as Model B, but opposite direction. These initial configurations are shown in the top to bottom panels of Fig. 1. The coefficient  $b_1$  or  $b_2$  is chosen for the maximum of  $|B_\phi|$  to be equal to 1, since the magnetic fields in the numerical calculation are normalized by the maximum value at the initial state. The magnetized parameter for these models is  $\mathcal{R}_m = 100$ .

The top row shows the evolution of  $l = 1$  with  $b_1 > 0$  (Model A). Snapshots of the configuration are shown at the initial state (left), at the Hall drift timescale  $\sim 6 \times 10^{-4} \tau_d$  (center) and at the Ohmic decay timescale  $\sim 2 \times 10^{-3} \tau_d$  (right). Note that the amplitude of  $s_1$  decays by  $\exp(-166t/\tau_d)$  in the absence of the Hall term (Hollerbach & Rüdiger 2002), so the typical decay time is not  $\tau_d$  but  $6 \times 10^{-3} \tau_d$ . The difference mainly comes from the choice of normalization length. The stellar radius is used in this paper to compare previous works, but more appropriate one is the crust size. Thus,  $\tau_d$  is rather large to characterize the actual Ohmic decay. The maximum of  $S$  at  $t = 0$  is located at  $r \approx (r_1 + r_s)/2$  and  $\theta = \pi/2$ , and moves in the negative  $Z$ -direction, as shown in close-ups in Fig.1. The drift stops at the outer boundary in several times the drift timescale. It is clear that the function  $S$  at this time contains higher multipole components  $s_l(r, t)$  in addition to the initial  $s_1(r, t)$  when it is expanded as in eq. (10). Subsequently the shape is nearly fixed, but overall amplitude gradually decreases with a longer Ohmic decay timescale. Some numerical calculations were performed longer until  $\sim \tau_d$ , but the behavior after  $6 \times 10^{-2} \tau_d$  is well described by a simple exponential decay. The results are limited to the early phase since subsequent evolution is easily inferred from the extrapolation.

The middle row shows the evolution of Model B ( $b_2 > 0$ ), and the bottom row shows the evolution of Model C ( $b_2 < 0$ ). The evolution of the magnetic configuration is quite different according to the initial sign of  $b_2$ . The positive and negative regions of  $S$  'collide' at the equatorial plane in Model B, while they 'repulse' each other in Model C. This occurs because the drift is negative in the  $Z$ -direction for  $S > 0$ , but positive in the  $Z$ -direction for  $S < 0$ . It is also important to note that the shape predominantly moves in not the  $\theta$ -direction, but the  $Z$ -direction (see eq. (8)). The motion in  $S$  is clear in close-ups of steep structure in Fig.1. In any model, the moved shape decays in a longer decay timescale.

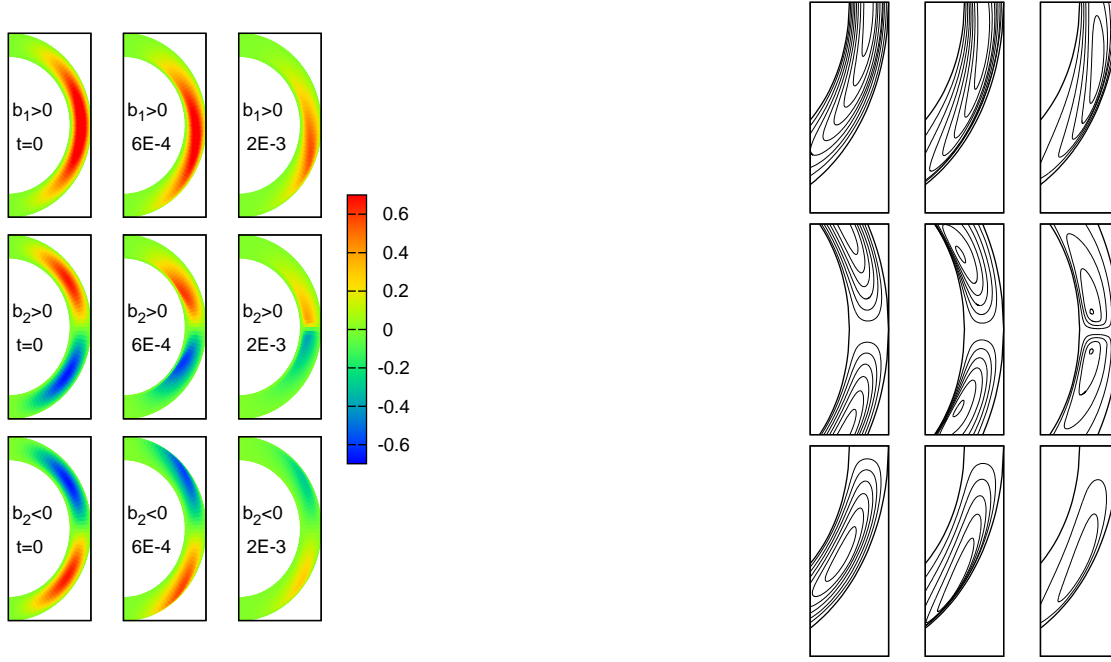
These results remarkably show the nature of the Hall drift, where a different initial sign for  $B_\phi$  leads to a different fate. This leads to an interesting question: Do these different configurations lead to significantly different dissipation rates? If so, the direction of  $B_\phi$  would have a significant effect. Figure 2 shows magnetic energy time evolutions for three models. The difference between Model B and Model C is relatively minor. A more marked difference is shown for the initial multipole. That is, the energies for Model B and C decay slightly faster than that of Model A. The current is swept to the outer boundary in Model C, but the outer boundary is replaced by the equatorial plane in Model B. In Fig. 2, a curve with  $\exp(-332t/\tau_d)$ , corresponding to the energy decay in the absence of the Hall drift, is also plotted for comparison.

Figure 3 shows the dependence of the magnetic parameter  $\mathcal{R}_m$ . The initial configuration is the same as that of Model B, the model of  $l = 2$  with  $b_2 > 0$ . The same figure also shows the magnetic energy for free decay. The damping is clearly strong for large  $\mathcal{R}_m$ . The Ohmic decay is enhanced in the presence of the Hall drift, that is, current is swept into a certain region, where it is effectively dissipated.

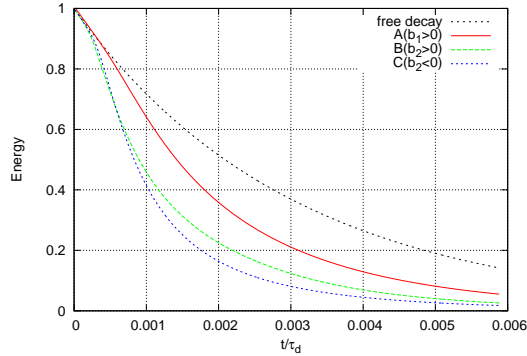
### 4 EVOLUTION OF MIXED FIELDS

In this section, field evolution is numerically studied for a mixed magnetic configuration consisting of poloidal and toroidal fields. The initial configuration is given solely by the  $l = 1$  component for both fields, namely, eq. (13) for  $s_1$  and eq. (14) for  $g_1$ . The maximum of each field is chosen as the same amplitude, and the magnetized parameter is  $\mathcal{R}_m = 100$ . Figure 4 shows snapshots of the evolving fields at representative times. The color contour represents the function  $S$  of the toroidal field, and lines denote the contour of the magnetic flux function  $G$  of the poloidal field.

Oscillatory behavior is clearly evident in  $G$ . Initially, the function decreases with the increase in cylindrical distance, and



**Figure 1.** Snapshots of time evolution for the function  $S$ , Model A ( $b_1 > 0$ ) at top, Model B ( $b_2 > 0$ ) in the middle, and Model C ( $b_2 < 0$ ) at bottom. Column times are  $t/\tau_d = 0, 6 \times 10^{-4}, 2 \times 10^{-3}$  from left to right. Whole spherical regions are shown by color contour in the left nine panels, while their close-ups of the steep region are shown by contour line with increment of 0.1 in the right nine panels. The region is limited to  $0.5 \leq R/r_s \leq 1, -1 \leq Z/r_s \leq 0$  in Model A,  $0.5 \leq R/r_s \leq 1, -0.5 \leq Z/r_s \leq 0.5$  in Model B, and  $0.5 \leq R/r_s \leq 1, -1 \leq Z/r_s \leq 0$  in Model C.

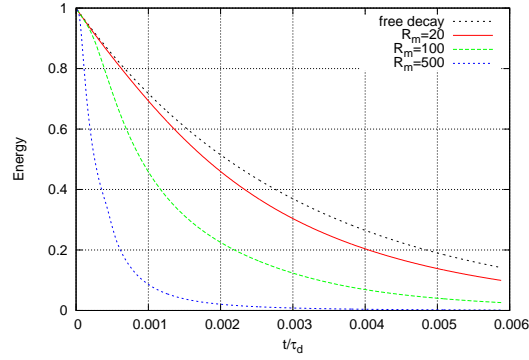


**Figure 2.** Normalized magnetic energy as a function of time  $t/\tau_d$  for three models with magnetization parameter  $\mathcal{R}_m = 100$ . Free decay curve is also plotted for comparison.

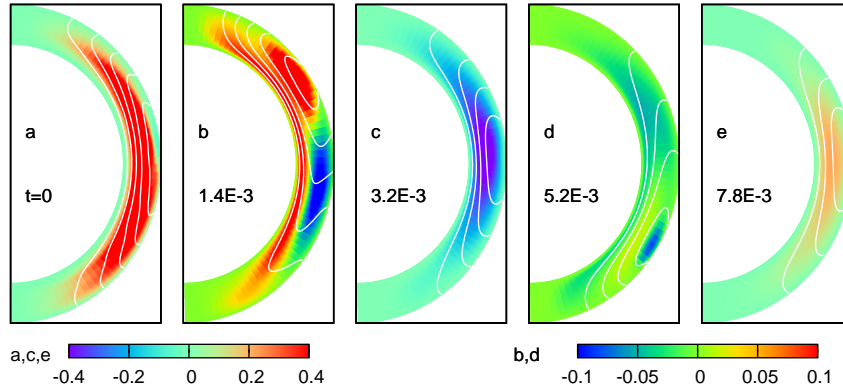
the maximum is located on the equator  $\theta = \pi/2$ . The maximum moves ‘upward’ in the meridian plane, toward  $\theta < \pi/2$ , until  $t/\tau_d \approx 1.4 \times 10^{-3}$  (second panel). It then changes direction and goes ‘downward,’ passing through the equator at  $t/\tau_d \approx 3.2 \times 10^{-3}$  (third panel) and reaching a minimum at  $t/\tau_d \approx 5.2 \times 10^{-3}$  (fourth panel), before returning to the initial position at  $t/\tau_d \approx 7.8 \times 10^{-3}$  (fifth panel). During this cycle, the field strength decreases.

The function  $S$  is also oscillatory. The initial configuration contains only the  $l = 1$  component in the angular part ( $S \propto \sin^2 \theta$ ), which is symmetric with respect to  $\theta = \pi/2$ . The state at  $t/\tau_d \approx 1.4 \times 10^{-3}$  (second panel) markedly differs from the initial state. The configuration is no longer symmetric, and higher multipoles can be seen. The field strength itself is weak around this time. At  $t/\tau_d \approx 3.2 \times 10^{-3}$  (third panel), the configuration again becomes symmetric like the initial state, but the sign of  $S$  is reversed. The  $l = 1$  component is dominated there. After the direction of  $B_\phi (= S/(r \sin \theta))$  again changes, the configuration returns to the initial one at  $t/\tau_d \approx 7.8 \times 10^{-3}$  (fifth panel). The directional change occurs around  $t/\tau_d \approx 1.4 \times 10^{-3}$  (second panel) and  $5.2 \times 10^{-3}$  (fourth panel), which correspond to a local minimum of toroidal field strength. The overall toroidal field strength also decreases during this cycle.

Figure 5 clearly shows the oscillatory behavior of the magnetic energy, which is divided into poloidal and toroidal parts,



**Figure 3.** Normalized magnetic energy as a function of time  $t/\tau_d$  for  $\mathcal{R}_m = 20, 100, 500$ . Free decay curve is also plotted for comparison.

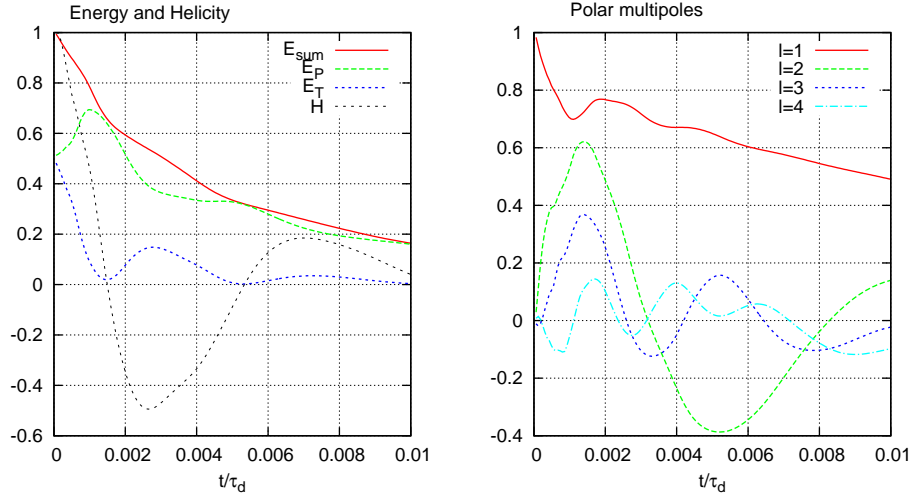


**Figure 4.** Snapshots of time evolution for functions  $G$  and  $S$ . Contour lines outwardly represent the level of  $G$  for  $0.02 \times n \times (B_0 r_s^3)$ ,  $n = 1, 2, \dots$ . Color contour represents  $S$  normalized by  $B_0 r_s$ . Note that different color scales are used, since  $S$  becomes very small at the bounces in the second and fourth panels. Those panels use the color scale on the left; the others, the scale on the right.

$E_P$  and  $E_T$ , respectively. The magnitude of the amplitudes at  $t = 0$  are nearly the same.<sup>1</sup> The curve of toroidal energy  $E_T$  represents a damped oscillation. Local minima can be seen at  $t/\tau_d \approx 1.4 \times 10^{-3}$  and  $5.2 \times 10^{-3}$ , corresponding to the time of the second and fourth panels in Fig. 4. The configurations of the third and fifth panels in Fig. 4 are those of the local maxima at  $t/\tau_d \approx 3.2 \times 10^{-3}$  and  $7.8 \times 10^{-3}$ . There is remarkable energy transfer between the toroidal and poloidal parts during the initial phase. Initially  $E_P$  increases until  $t/\tau_d \approx 1.4 \times 10^{-3}$ , although the total energy  $E_{sum} = E_P + E_T$  decreases. The sum always decreases due to Ohmic decay (see eq. (18)). The initial rapid decay of  $E_T$  is thus partially due to this transfer. Energy is subsequently transferred between the two components in turn, but the behavior becomes less clear. The magnetic field decays on the timescale  $t/\tau_d \approx 10^{-2}$ , so the coupling becomes weak. The oscillation period gradually becomes longer, since the drift timescale increases. Figure 5 also shows the evolution of magnetic helicity  $H$ , which exhibits oscillatory damping with Ohmic decay timescale  $t/\tau_d \approx 10^{-2}$ . The change of sign in  $H$  denotes an inversion of the toroidal field  $B_\phi$ , which occurs around the local minima of  $E_T$ .

The coefficients  $a_l$  of the multi-moments in eq. (12) describe the exterior poloidal field. The right panel of Fig. 5 shows the evolution of a few of the lowest values. Higher multipoles are induced until  $t/\tau_d \approx 1.4 \times 10^{-3}$ , the configuration shown in

<sup>1</sup> Values are not exactly the same, because the maximum amplitude of each field is fixed at same value, the distributions are slightly different.



**Figure 5.** Time evolution of energy  $E$ , helicity  $H$  (left panel), and multipole moments  $a_l$  at the surface (right panel). Energy is normalized by the initial total energy, and helicity by the initial value. Coefficient  $a_l$  of the  $l$ -th moment is normalized by the initial dipole value  $a_1 r_s^{l-1}$ .

the second panel of Fig. 4. The poloidal field is no longer dipole at this time. Interestingly, the coefficient  $a_1$  is significantly decreased, as compared with the initial value, although a large amount of energy is stored in the poloidal part, as shown in the left panel of Fig. 5. Dipole field strength at this time is not a good indicator of overall magnetic energy. The poloidal field returns back to the dipole at  $t/\tau_d \approx 3.2 \times 10^{-3}$  (the time of the third panel in Fig. 4), at which higher multipoles are temporarily zero. After sinusoidal oscillation, higher multipoles decay rather rapidly, so the polar dipole remains for the later time  $t/\tau_d > 10^{-2}$ . The behavior after  $10^{-2}\tau_d$  is rather simple. Magnetic energy is significantly dissipated, so that the Hall drift becomes less important. Subsequent evolution in the late phase is described by a free decay due to Ohmic dissipation, and is omitted here.

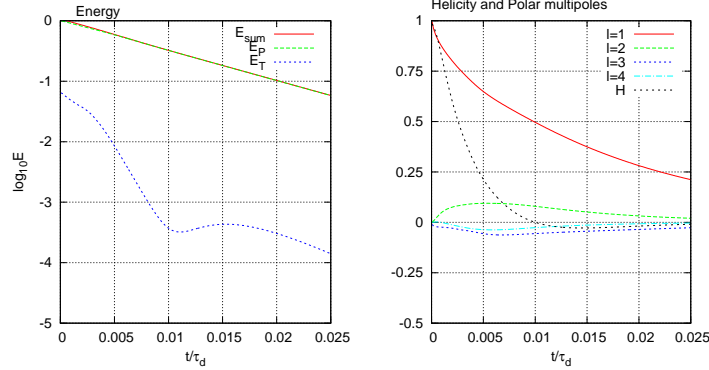
Figures 6–9 compare evolutions of energy, helicity, and polar multipole coefficients for four models. The initial magnetic configuration is the same as that of Figs. 4 and 5, but the strength and magnetized parameter  $\mathcal{R}_m$  vary. The ratio of poloidal to toroidal fields is 1/4, or 4 in amplitude, so  $16(=4^2)$  or  $1/16$  in energy, and  $\mathcal{R}_m$  is  $\mathcal{R}_m = 20$ , or 100. The right panel shows energy plotted on a logarithmic scale,  $\log_{10}(E)$ , and the left helicity  $H$  and a few lowest multipole coefficients  $a_l$ . Figure 6 shows the results of a model with  $\mathcal{R}_m = 20$ , in which the energy is initially dominated by the poloidal part. Polar multipole components are induced through the coupling to the toroidal part, but the dipole field is barely affected. The decay curve is well described by  $\exp(-55t/\tau_d)$ , which coincides with free decay of the dipole. The energy is always dominated by the polar dipole, and the evolution is described by  $\exp(-110t/\tau_d)$ , a line in the logarithmic scale. The curve of toroidal energy is nonlinear on the logarithmic scale, but is oscillatory. The minor component is highly affected by the poloidal one, but is almost neglected in the dynamics of the whole system. No consequence of the toroidal field comes from the fast decay rate  $\exp(-166t/\tau_d)$  in the amplitude as seen in Section 4, or from the initial small strength.

Figure 7 shows the results of an evolution in which the toroidal component is dominated at the initial state. Compared with the decay curve of total energy in Fig. 6, the damping is much faster. Most of the magnetic energy is initially stored in the toroidal part, but rapidly decays. The poloidal component decays rather slowly, and dominates at the later time. The higher multipole moments are induced and have larger amplitudes than those of Fig. 6. The overall dipole decay curve is nonetheless very similar to that of the large poloidal case.

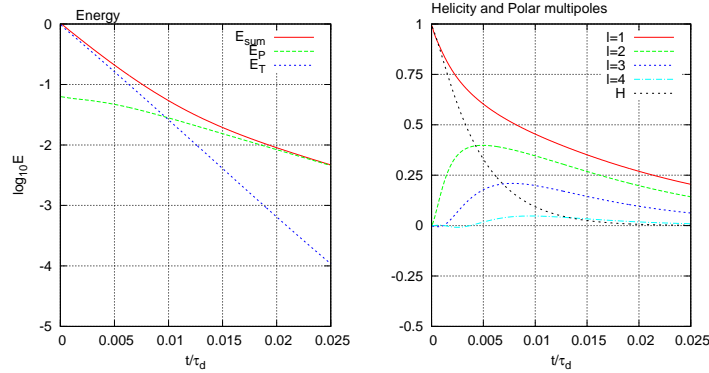
Figure 8 shows the results for large a magnetization parameter  $\mathcal{R}_m = 100$  and an initially large poloidal field. The evolution of the total energy is very similar to that of Fig. 6, although the oscillatory behavior in the toroidal energy is evident. The timescale determined by the Hall drift is approximately 1/5 that of Fig. 6, in which  $\mathcal{R}_m = 20$ . The oscillatory behavior is also clear in the coefficients  $a_l$  and helicity  $H$ . However, except for the initial wavy structure seen in the Hall drift timescale, the decay of the dipole field is similar to that of Fig. 6. The dominant component, the polar dipole, is thus not affected by the toroidal field, which never plays an important role because of its rapid decay.

Figure 9 shows the results for a large magnetization parameter  $\mathcal{R}_m = 100$  and an initially large toroidal field. In this case,





**Figure 6.** Time evolution of energy in logarithmic scale  $\log_{10}(E)$  (left panel) and helicity  $H$  and multipole moments  $a_l$  at the surface (right panel) for an initially large poloidal field with  $\mathcal{R}_m = 20$ . These normalizations are the same as in Fig. 5.

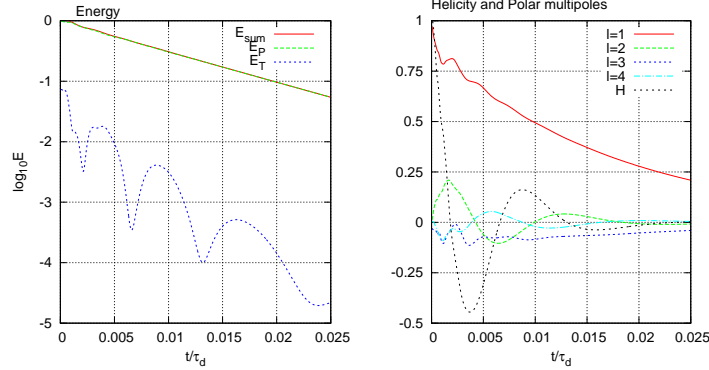


**Figure 7.** Same as Fig. 6, but for an initially large toroidal field with  $\mathcal{R}_m = 20$ .

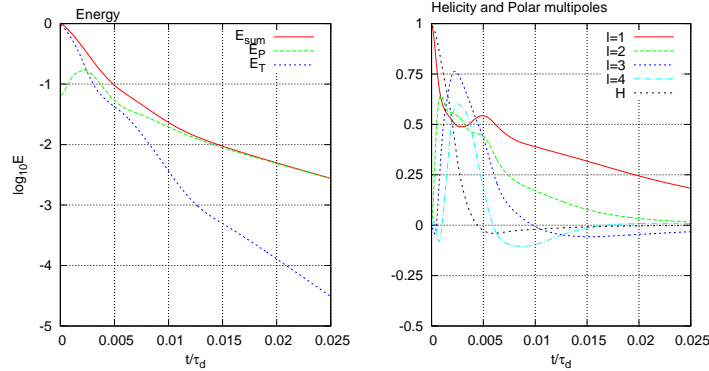
a large amount of energy is transferred from the toroidal to poloidal components. Higher multipoles in the poloidal field are induced around  $t/\tau_d \approx 10^{-3}$ . The configuration of the poloidal field at this time is significantly distorted as in the second panel in Fig. 4. The toroidal energy becomes much smaller than the poloidal energy around  $t/\tau_d \approx 10^{-2}$ , and both components are subsequently decoupled. After that, the dipole field evolution is determined by free decay. Despite the disorder at  $t/\tau_d \approx 10^{-3}$ , the amplitude of the free decay phase, for example at  $t/\tau_d = 2 \times 10^{-2}$ , does not differ so much from that of Figs. 6–8.

## 5 SUMMARY AND DISCUSSION

Numerical simulations demonstrate how the Hall drift changes the current and magnetic configuration. In a purely toroidal evolution, Ohmic dissipation is enhanced by accumulated currents elsewhere. The spatial location depends on the initial data, but the energy dissipation rate does not so significantly depend on the accumulation position in our uniform conductivity model. In a realistic case the conductivity will decrease with the radius, so most of the magnetic energy may be effectively dissipated near the surface. The results become more sensitive to the initial condition. Energy is, in principle, transferred between poloidal and toroidal components if both are initially involved. It is important for understanding the evolution that the free decay rate of the polar dipole is the least. When the polar dipole dominates, it is rarely affected by the toroidal field, which rapidly decays. On the other hand, a significant amount of energy is transferred to the poloidal field until almost equipartition when the toroidal component is initially dominant. Global nonlinear coupling is manifest in the Hall drift timescale only when the corresponding energies are of the same order. Moreover, the poloidal field at the surface and the



**Figure 8.** Same as Fig. 6, but for an initially large poloidal field with  $\mathcal{R}_m = 100$ .



**Figure 9.** Same as Fig. 6, but for an initially large toroidal field with  $\mathcal{R}_m = 100$ .

exterior is highly distorted, and is no longer described by a pure dipole field. During this phase, the dipole is not a good indicator of overall field strength. In a longer Ohmic dissipation timescale the toroidal field decays rapidly, so the coupling vanishes. The polar dipole eventually survives.

The early evolution highly depends on the choice of the initial data, i.e., the configuration and strength. As discussed by Reisenegger et al. (2007), stationary conditions in the presence of Hall drift for an axisymmetric magnetic field means that the isosurface of  $S$  coincides with that of  $G$ ; in other words,  $S = S(G)$ , for which the coupling term  $\vec{\nabla}G \times \vec{\nabla}S$  vanishes. This condition is not satisfied in the whole shell region. The outer boundary condition requires that the toroidal component is concentrated in the interior ( $S = 0$ ), whereas the poloidal one may leak out to the exterior, meaning that  $G \neq 0$  at the surface. The interior poloidal field is described by the value at the surface if the topology is simply connected. The condition  $S(G)$  therefore means that  $S = 0$  everywhere, irrespective of  $G$ . Our numerical calculation starts with an initial configuration different from the ‘Hall equilibrium’, and shows the behavior to the state  $S = 0$ . It may be necessary as a next step to study plausible initial configurations, since little known about them.

The timescale is less accurate in our simplified model with uniform density and conductivity. Realistic models are necessary, but the distributions depend on equation of state and neutron star mass. Assuming that our uniform model is obtained as a result of spatial average of a certain model with stratified number density and conductivity, the timescale is estimated. The overall normalization constant  $\tau_d$  is given by  $\tau_d = 4\pi\sigma r_s^2/c^2 \approx 4.4 \times 10^9 (\bar{\sigma}/10^{25}\text{s}^{-1}) (r_s/10\text{km})^2$  years, where  $\bar{\sigma}$  is an averaged value of the conductivity. The decay timescale of the dipole is  $2 \times 10^{-2}\tau_d \approx 9 \times 10^7$  years, and that of the toroidal one is  $6 \times 10^{-3}\tau_d \approx 3 \times 10^7$  years. These numbers are slightly larger, since the crust size in our present model is thick,  $L = r_s/4$ .

The timescale is proportional to  $L^2$ , so that the actual values may be smaller by a factor of  $\sim 10^{-2}$ – $10^{-1}$ . A typical decay timescale is around  $10^5$ – $10^7$  years.

The characteristic age of the low-field magnetar SGR0418+5729 is more than  $2.4 \times 10^7$  years (Rea et al. 2010). The dipole field ( $< 7.5 \times 10^{12}$  G) may decay or survive within this period. However, the internal toroidal field is likely to dissipate more quickly. It is therefore difficult to understand why the toroidal field has a much larger field strength,  $> 10^{13}$ – $10^{14}$  G, which is required for the activity. One possible explanation is that current age estimates are inaccurate. The characteristic age is normally estimated by assuming a constant dipole magnetic field. The Hall drift significantly affects the surface value, especially at the initial epoch with strong field. As demonstrated in the second panel of Fig. 4, the surface field is highly distorted from the pure dipole. SGR0418+5729 may correspond to a young phase of oscillatory evolution in which the surface dipole temporarily decreases, but there is a strong internal toroidal component.

## ACKNOWLEDGEMENTS

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